J. Fluid Mech. (1973), *vol.* 61, *part* 1, *pp.* 73–84
Printed in Grea Britain

Resonant oscillations in radiative magnetogasdynamics

By J. B. HELLIWELL
School of Mathematics, University of Bradford, Yorkshire

(Received 30 April 1973)

Investigations are made of the wave motion which arises near resonance in a tube with an applied transverse magnetic field filled with a highly electrically conducting gas and closed by two rigid walls. The wave motion is driven by the sinusoidal radiative flux emitted by one of the walls as a consequence of its oscillatory temperature; the other wall is taken to be a perfect reflector of thermal radiation. The effects of radiative transfer are treated by the use of the differential approximation. The analysis leads to the same formal governing integral equation for the solution as arises in the ordinary gasdynamic case in the absence of electromagnetic effects. Within a narrow frequency band around resonance the theory predicts the occurrence of magnetogasdynamic shock waves which become dispersed as the thermal radiation is strengthened and may be totally dispersed to leave a continuous, periodic, but not necessarily sinusoidal, wave motion. The effect of the magnetic field is to delay the onset of dispersion.

1. Introduction

This paper is devoted to a study of the oscillations near resonance which arise in a perfectly electrically conducting inviscid gas confined within a closed tube with a transverse applied magnetic field. The disturbances are supposed to be generated by a small sinusoidal variation of the temperature of the rigid wall at one end of the tube, which thus emits a similarly varying radiative flux into the gas. The other end of the tube is closed by another rigid wall, which is a perfect reflector of thermal radiation. The wave motion which develops is thus dependent upon the interaction of effects of thermal radiation, in particular gas emission, with the magneto-acoustic phenomena which are stimulated by the temperature and pressure changes brought about by the absorption of the radiative energy.

Resonant oscillations in gasdynamics have been studied for many years but the antecedents of the present work need only be traced back to the investigations of Chester (1964). In strictly linear acoustic theory of one-dimensional non-radiative wave motion in a closed tube a demand for infinite amplitude at resonance leads to a breakdown of the analysis there and dubious validity of the results near resonance. Experiment shows that, in a narrow frequency band around each resonant frequency, shock waves appear in the tube. Chester showed, by deductive reasoning, that the retention of certain nonlinearities in the formulation for inviscid flow near resonance leads to a solution valid for all frequencies and that shock waves arise in a natural manner in this solution.

Similar difficulties arise in the study of radiative acoustics. Long & Vincenti (1967) showed, on the basis of a linearized inviscid analysis, that at certain frequencies the amplitude of the oscillations becomes much greater than that elsewhere but remains finite as a consequence of the weak radiative damping present owing to the perturbations in the radiative emission from the gas. The validity of the theory is thus questionable near these frequencies. Subsequently Eninger & Vincenti (1973) applied the methods of Chester to the analysis of radiatively driven wave motion in a closed tube and again showed how, near resonance, the small perturbation assumption of a strictly linear theory is violated. They assumed a fairly large Boltzmann number ($\sim$400–100000), appropriate to experimental conditions, and showed that shock waves could arise in the tube, but that as the effects of thermal radiation became more significant the shocks were dispersed and for sufficiently strong effects would vanish leaving a continuous solution.

In the present investigation the problem studied by Eninger & Vincenti is extended to include electromagnetic effects when the gas in the tube has a high electrical conductivity. Whilst these authors employed the exponential approximation to simplify the exact integro-differential expressions for the contribution from the thermal radiation, here an analysis is given of the problem using the differential approximation. In order that the consequences of the two approaches may be compared the final results of an analysis using the exponential approximation are also presented. In view of the already known close correspondence between the two approximations, see, for instance, Vincenti & Kruger (1965), it is not surprising that these are formally very similar. The wave pattern in the tube depends upon the solution of a certain integral equation which in the case of the exponential approximation may be reduced by suitable transformations to become identical with that discussed by Eninger & Vincenti. The interaction between thermal radiation and magneto-acoustic waves in a tube at all conditions including those near resonance is thus similar to that in the absence of electromagnetic effects.

2. Basic equations

Consider the unsteady one-dimensional flow of a grey gas of infinite electrical conductivity and absorption coefficient α^* parallel to the x^* axis in a domain bounded by rigid planes at $x^* = 0$ and $x^* = L$, in the presence of an applied magnetic field H_0^* parallel to the z^* axis of a Cartesian system. The gas is taken to be calorifically perfect, inviscid and sufficiently hot for effects of thermal radiation to be significant. The wall at $x^* = 0$ is supposed to radiate as a black body and to have a temperature T_w^* which oscillates sinusoidally about a mean value T_0^* so that

$$T_w^* = T_0^*(1 + T_w \sin \omega t), \tag{2.1}$$

with $T_w \ll 1$. The wall at $x^* = L$ is supposed to be a perfect reflector of thermal radiation.

The governing equations are those of magnetogasdynamics as given, for instance, by Cabannes (1970, chap. 1), suitably modified by the inclusion of a contribution from the radiative flux in the equation of energy conservation, together with appropriate equations to express the transfer of thermal radiation. The former of these may be written as

$$\frac{\partial H^*}{\partial t^*}+\frac{\partial}{\partial x^*}(u^*H^*) = 0, \quad \frac{\partial \rho^*}{\partial t^*}+\frac{\partial}{\partial x^*}(u^*\rho^*) = 0, \tag{2.2), (2.3}$$

$$\rho^*\left(\frac{\partial u^*}{\partial t^*}+u^*\frac{\partial u^*}{\partial x^*}\right)+\mu H^*\frac{\partial H^*}{\partial x^*}+\frac{\partial p^*}{\partial x^*} = 0, \tag{2.4}$$

$$c_v\rho^*\left(\frac{\partial T^*}{\partial t^*}+u^*\frac{\partial T^*}{\partial x^*}\right)+p^*\frac{\partial u^*}{\partial x^*}+\frac{\partial q^*}{\partial x^*} = 0, \tag{2.5}$$

$$p^* = R\rho^*T^* = \rho^{*\gamma}\exp\{(s^*-s_b^*)/c_v\}, \tag{2.6}$$

where $u^*, p^*, \rho^*, H^*, T^*, q^*$ and s^* denote respectively the velocity, pressure, density, magnetic field strength, temperature, radiative flux and specific entropy, s_b^* denotes a base level of entropy, γ, c_v and μ are respectively the adiabatic index, specific heat at constant volume and permeability of the gas and t^* is the time.

It is well known that an exact statement of the equation of radiative transfer and its relationship to the radiative flux gives rise to a coupled system of integro-differential equations. Since these lead to intractable analysis various forms of approximation have been employed to simplify the formulation and to generate equations which are entirely differential. Here we employ the form known as the differential approximation, which has been shown to be particularly appropriate for use with one-dimensional problems, see for instance Vincenti & Kruger (1965, chap. 12). In terms of the forward and backward flux components, $q_{\mp}^*$, in a gas of general opacity this may be written, see, for example, Helliwell (1966), as follows:

$$q^* = q_-^* - q_+^*, \tag{2.7}$$

$$\partial(q_-^* - q_+^*)/\partial x^* = 4\sigma\alpha^*T^{*4} - 2\alpha^*(q_-^* + q_+^*), \tag{2.8}$$

$$\partial(q_-^* + q_+^*)/\partial x^* = -\tfrac{3}{2}\alpha^*(q_-^* - q_+^*), \tag{2.9}$$

where α^* is the volumetric absorption coefficient and σ is Stefan's constant.

The boundary conditions are given by

$$u^* = 0 \quad \text{at} \quad x^* = 0, L, \tag{2.10}$$

$$q_-^* = \begin{cases} \sigma T_w^{*4} & \text{at} \quad x^* = 0, \quad (2.11) \\ q_+^* & \text{at} \quad x^* = L. \quad (2.12) \end{cases}$$

Now introduce generally the suffix zero to indicate the mean state of the gas and express the general state in terms of perturbations about this mean state. Thus, since the frozen magnetic flux relationship that H^* and ρ^* are proportional is established from (2.2) and (2.3), we write

$$H^*/H_0^* = \rho^*/\rho_0^*. \tag{2.13}$$

Appropriate non-dimensional perturbation variables are defined as follows:

$$\left.\begin{aligned} p^* = p_0^*(1+p), \quad \rho^* = \rho_0^*(1+\rho), \quad T^* = T_0^*(1+T), \quad u^* = c_0^* u, \\ s^* - s_0^* = c_v s, \quad q_\mp^* = \sigma T_0^{*4} q_\mp, \quad q^* = \sigma T_0^{*4} q. \end{aligned}\right\} \tag{2.14}$$

Here c_0^* is the mean magneto-acoustic speed in the gas, given by

$$c_0^{*2} = a_0^{*2} + \mu H_0^{*2}/\rho_0^*, \tag{2.15}$$

where a_0^* is the mean acoustic speed. Similarly, dimensionless position and time variables are introduced by defining

$$x = x^*\omega/c_0^*, \quad t = \omega t^*. \tag{2.16}$$

The substitution of these new variables into the governing equations (2.1)–(2.12) leads to a system of dimensionless equations which contains a number of parameters. These are most conveniently taken to be the following:

$$\beta = \{1 + \mu H_0^{*2}/\rho_0^* a_0^{*2}\}^{-\frac{1}{2}}, \quad \text{the Alfvén number,} \tag{2.17}$$

$$B = c_p \rho_0^* c_0^*/\sigma T_0^{*3}, \quad \text{the Boltzmann number,} \tag{2.18}$$

$$D = \alpha_0^* c_0^*/\omega, \quad \text{the Bouguer number,} \tag{2.19}$$

$$l = L\omega/c_0^*, \quad \text{the resonance number,} \tag{2.20}$$

where c_p is the specific heat at constant pressure. As already remarked, in laboratory conditions in which mechanical processes of energy transfer dominate those of radiation the Boltzmann number is large; thus as in the earlier work of Long & Vincenti (1967) and Eninger & Vincenti (1973) the further theory is developed on the assumption that $1/B \ll 1$.

Equations (2.2)–(2.9) are now expanded and terms up to and including those of second order in the perturbation variables are retained. From (2.6) the temperature and pressure perturbation may be expressed in terms of the entropy and density variations. Thus

$$p = \gamma\rho + s + \tfrac{1}{2}\gamma(\gamma-1)\rho^2 + \gamma\rho s + \tfrac{1}{2}s^2, \tag{2.21}$$

$$T = (\gamma-1)\rho + s + \tfrac{1}{2}(\gamma-1)(\gamma-2)\rho^2 + (\gamma-1)\rho s + \tfrac{1}{2}s^2. \tag{2.22}$$

The remaining equations yield

$$\frac{\partial\rho}{\partial t} + \frac{\partial u}{\partial x} = -\rho\frac{\partial u}{\partial x} - u\frac{\partial\rho}{\partial x}, \tag{2.23}$$

$$\frac{\partial u}{\partial t} + \frac{\partial\rho}{\partial x} + \frac{\beta^2}{\gamma}\frac{\partial s}{\partial x} = -u\frac{\partial u}{\partial x} - \frac{\beta^2}{\gamma}\left\{\gamma[(\gamma-2)\rho + s]\frac{\partial\rho}{\partial x} + [(\gamma-1)\rho + \tfrac{1}{2}s]\frac{\partial s}{\partial x}\right\}, \tag{2.24}$$

$$\frac{\partial s}{\partial t} + \frac{\gamma}{B}\frac{\partial}{\partial x}(q_- - q_+) = -u\frac{\partial s}{\partial x} + \frac{\gamma}{B}(\gamma\rho + s)\frac{\partial}{\partial x}(q_- - q_+), \tag{2.25}$$

$$\begin{aligned} \partial(q_- - q_+)/\partial x + 2D(q_- + q_+) - 16D\{(\gamma-1)\rho + s\} = 8D\{(\gamma-1)(4\gamma-5)\rho^2 \\ + 8(\gamma-1)\rho s + 4s^2\}, \end{aligned} \tag{2.26}$$

$$\partial(q_- + q_+)/\partial x + \tfrac{3}{2}D(q_- - q_+) = 0. \tag{2.27}$$

It is sufficient for our purposes to retain a fully linearized form for the boundary conditions, which therefore for all t are such that

$$\left.\begin{aligned} q_- = 4T_w \sin t, \quad u = 0 \quad \text{at} \quad x = 0, \\ q_- = q_+, \quad u = 0 \quad \text{at} \quad x = l. \end{aligned}\right\} \tag{2.28}$$

3. The simplified linear problem

Consider first the linear problem, for which, since the Boltzmann number is large, it is adequate to neglect the contributions to (2.25) which arise from the terms in (2.26) relating to the perturbations in the gas emission. Equations (2.26) and (2.27) then separate from the remainder and together with the radiative boundary conditions (2.28) may be solved independently for $q_\mp$. Whence using (2.25) to determine the entropy and substituting into (2.23) and (2.24), the fully linearized forms of the latter may be expressed as the pair of inhomogeneous wave equations

$$\left(\frac{\partial}{\partial t} \pm \frac{\partial}{\partial x}\right)(u \pm \rho) = -\frac{48\beta^2 D^2}{B}\left\{\frac{T_w \cos t}{2 + 3^{\frac{1}{2}} - (2 - 3^{\frac{1}{2}})\exp(-2 \times 3^{\frac{1}{2}} Dl)}\right\}$$
$$\times \{\exp(-3^{\frac{1}{2}} Dx) - \exp[-3^{\frac{1}{2}} D(2l - x)]\}. \tag{3.1}$$

These may be solved without undue difficulty. A complementary function which satisfies the condition $u = 0$ at $x = 0$ may be written as

$$u \pm \rho = \pm 2f(t \mp x). \tag{3.2}$$

A particular integral satisfying the same condition is derived and finally the form of the function f is obtained by fitting the complete solution to the remaining boundary condition (2.28). One finds the final solution

$$u \pm \rho = \frac{48\beta^2 D^2}{B(1 + 3D^2)}\left\{\frac{T_w}{2 + 3^{\frac{1}{2}} - (2 - 3^{\frac{1}{2}})\exp(-2 \times 3^{\frac{1}{2}} Dl)}\right\}\left\{\mp \frac{1 - \exp(-2 \times 3^{\frac{1}{2}} Dl)}{\tan l}\right.$$
$$\times \cos(t \mp x) + \exp(-3^{\frac{1}{2}} Dx)(\pm 3^{\frac{1}{2}} D \cos t - \sin t)$$
$$+ \exp[-3^{\frac{1}{2}} D(2l - x)](\pm 3^{\frac{1}{2}} D \cos t + \sin t)$$
$$\left. + [1 - \exp(-2 \times 3^{\frac{1}{2}} Dl)] \sin(t \mp x)\right\}, \tag{3.3}$$

in which the first term in braces is the complementary function and the remaining three terms compose the particular integral.

Whilst the linear theory and solution are valid in most circumstances, exactly at resonance the amplitude of the complementary function increases without bound as $\tan l$ falls to zero, but the particular integral remains finite throughout. The difficulty arises when a node in the complementary function for u, at resonance, coincides with the position of the reflecting wall. Under these circumstances no finite amplitude could suffice to adjust the complementary function to fit, in conjunction with the particular integral, the boundary condition at this wall. Thus the linear small perturbation theory must become invalid near resonance, when $l \approx N\pi$, where N is an integer; nonlinear terms are significant and must be taken into consideration.

4. A uniformly valid approximation near resonance

Since near resonance it is essential to include at least some of the effects of nonlinearity, by implication it is likewise necessary to take into account the perturbations in emission from the gas. On the basis of a simplified linear theory it has been seen that the amplitude of the complementary function is much larger than that of the particular integral. Now a possible approach to the nonlinear problem is to seek an iterative solution as Chester (1964) did in his study of a related problem. He showed how the failure of the approximate solution could be avoided by delaying application of the boundary condition at the reflecting wall until one has derived the first iterated solution, which in view of the above remark may be obtained by iterating upon the complementary function of the simplified linear theory alone. The approximate solution so obtained is thus uniformly valid whether or not the tube is near resonance. Before, however, embarking upon the further analysis it is necessary to examine which of the terms previously omitted from (2.23)–(2.27) should be retained as significant.

The radiation from the black wall which drives the oscillations enters the solution via the boundary conditions on $q_{\pm}$ applied to (2.26) and (2.27) and ultimately gives rise to the particular integral. Clearly, then, to leading order $q_{\pm}$ have the same order as these driving perturbations and by way of (2.25) so does the entropy variation s. On the other hand the velocity u and density ρ contain terms from both the complementary function and the particular integral. Thus since near resonance the former dominates the latter it follows that the magnitudes of u and ρ are much larger than those of s and $q_{\pm}$ although all remain small. Hence whilst it is necessary to retain second-order terms in u and ρ it is sufficient to omit all but linear terms in s and $q_{\pm}$. Furthermore since B is large it is adequate to reject all but linear terms in (2.26) and of these to neglect the term in s compared with that in ρ.

Therefore the appropriate nonlinear equations containing terms of leading order are

$$\frac{\partial \rho}{\partial t}+\frac{\partial u}{\partial x} = -\rho\frac{\partial u}{\partial x}-u\frac{\partial \rho}{\partial x}, \tag{4.1}$$

$$\frac{\partial u}{\partial t}+\frac{\partial \rho}{\partial x}+\frac{\beta^2}{\gamma}\frac{\partial s}{\partial x} = -u\frac{\partial u}{\partial x}-\beta^2(\gamma-2)\rho\frac{\partial \rho}{\partial x}, \tag{4.2}$$

$$\frac{\partial s}{\partial t}+\frac{\gamma}{B}\frac{\partial}{\partial x}(q_- - q_+) = 0, \tag{4.3}$$

$$\partial(q_- - q_+)/\partial x + 2D(q_- + q_+) = 16D(\gamma-1)\rho, \tag{4.4}$$

$$\partial(q_- + q_+)/\partial x + \tfrac{3}{2}D(q_- - q_+) = 0, \tag{4.5}$$

subject to the boundary conditions (2.28).

Consider the last two equations (4.4) and (4.5). Iterating upon the complementary function (3.2) of the simplified linear theory, satisfying $u(0,t) = 0$, the insertion of the form

$$\rho = f(t-x)+f(t+x)$$

into the right-hand side of (4.4) leads to a pair of equations for $q_\pm$, the solution of which depends upon both the radiative boundary conditions (2.28) and the, meantime, arbitrary function f. On solving these equations one finds the form for the radiative flux

$$q_- - q_+ = \frac{16}{2+3^{\frac{1}{2}}-(2-3^{\frac{1}{2}})\exp(-2\times 3^{\frac{1}{2}}Dl)}\Big\{[\exp(-3^{\frac{1}{2}}Dx) - \exp\{-3^{\frac{1}{2}}D(2l-x)\}]\,T_w \sin t - 2(\gamma-1)\,D\exp(-3^{\frac{1}{2}}Dl)\{3^{\frac{1}{2}}\cosh(3^{\frac{1}{2}}Dx) + 2\sinh(3^{\frac{1}{2}}Dx)\}\int_0^l \cosh[3^{\frac{1}{2}}D(l-\eta)]\{f(t-\eta)+f(t+\eta)\}\,d\eta\Big\} + 16(\gamma-1)\,D\int_0^x \cosh[3^{\frac{1}{2}}D(x-\eta)]\{f(t-\eta)+f(t+\eta)\}\,d\eta. \tag{4.6}$$

The entropy perturbation is derived by substitution from (4.6) into (4.3) and is then inserted into (4.2). Also the forms for u and ρ obtained from (3.2) are substituted into the right-hand sides of (4.1) and (4.2), following which suitable combination and rearrangement leads to the pair of inhomogeneous wave equations

$$\left(\frac{\partial}{\partial t}\pm\frac{\partial}{\partial x}\right)(u\pm\rho) = \frac{-48\beta^2D^2}{B[2+3^{\frac{1}{2}}-(2-3^{\frac{1}{2}})\exp(-2\times 3^{\frac{1}{2}}Dl)]} \times\{\exp(-3^{\frac{1}{2}}Dx)-\exp[-3^{\frac{1}{2}}D(2l-x)]\}\,T_w\cos t - \frac{16(\gamma-1)\beta^2D}{B}\Big\{f(t-x)-f(t+x)+3D^2\exp[-3^{\frac{1}{2}}D(l-x)] \times\frac{2+3^{\frac{1}{2}}-(2-3^{\frac{1}{2}})\exp(-2\times 3^{\frac{1}{2}}Dx)}{2+3^{\frac{1}{2}}-(2-3^{\frac{1}{2}})\exp(-2\times 3^{\frac{1}{2}}Dl)} \times\int_0^l \cosh[3^{\frac{1}{2}}D(l-\eta)]\{\hat{f}(t-\eta)+\hat{f}(t+\eta)\}\,d\eta - 3D^2\int_0^x \cosh[3^{\frac{1}{2}}D(x-\eta)]\{\hat{f}(t-\eta)+\hat{f}(t+\eta)\}\,d\eta\Big\} + \{\pm[f'(t\mp x)\{3f(t\mp x)-f(t\pm x)\}+f'(t\pm x)\{f(t-x) + f(t+x)\}] + (\gamma-2)\,\beta^2[f(t-x)+f(t+x)]\,[f'(t-x)-f'(t+x)]\}, \tag{4.7}$$

where a prime denotes the derivative of a function with respect to its argument and

$$\hat{f}(t\pm\eta) = \int^t f(t\pm\eta)\,dt. \tag{4.8}$$

The terms on the right-hand side, in order, correspond respectively to the basic driving radiation from the black wall as in the simplified linear theory, see (3.1), the effects of perturbations in the gas emission and contributions from nonlinear magnetogasdynamic effects.

The solution of (4.7) is now required subject to the boundary conditions (2.28) upon the velocity. The form of the complementary function satisfying $u = 0$ at $x = 0$ is identical with that of the simplified linear theory and is expressed in (3.2). The various contributions to the particular integral from the above three components are derived in a straightforward manner but after protracted algebra. We consider them in turn.

The effects of the driving radiation are determined as in the simplified linear theory. As there, one seeks a particular integral of the form

$$u \pm \rho = \exp(-3^{\frac{1}{2}}Dx)F_{\pm}(t) + \exp\{-3^{\frac{1}{2}}D(2l-x)\}G_{\pm}(t) + H_{\pm}(t \mp x) \tag{4.9}$$

with the last term chosen so that the full expression satisfies the boundary condition $u(0,t) = 0$. One finds the expression given by the appropriate part of (3.3).

In deriving the contribution from the gas emission we follow the earlier analysis of Eninger & Vincenti. On expanding the hyperbolic functions in the relevant terms on the right-hand side of (4.7) into exponential functions one notes that a suitable expression for the particular integral may be written, formally, as

$$\begin{aligned} u \pm \rho = {} & F_{\pm}(t \mp x) + xJ_{\pm}(t \mp x) + K_{\pm}(t \pm x) \\ & + \int_0^x \{\exp[-3^{\frac{1}{2}}D(x-\eta)][G_{\pm}(t-\eta) + G_{\pm}(t+\eta)] + \exp[-3^{\frac{1}{2}}D(2l-x+\eta)] \\ & \times [\tilde{G}_{\pm}(t-\eta) + \tilde{G}_{\pm}(t+\eta)]\} d\eta \\ & + \int_x^l \{\exp[-3^{\frac{1}{2}}D(\eta-x)][H_{\pm}(t-\eta) + H_{\pm}(t+\eta)] + \exp[-3^{\frac{1}{2}}D(2l+x-\eta)] \\ & \times [\tilde{H}_{\pm}(t-\eta) + \tilde{H}_{\pm}(t+\eta)]\} d\eta \\ & + \int_0^l \{\exp[-3^{\frac{1}{2}}D(2l-x-\eta)][I_{\pm}(t-\eta) + I_{\pm}(t+\eta)] + \exp[-3^{\frac{1}{2}}D(x+\eta)] \\ & \times [\tilde{I}_{\pm}(t-\eta) + \tilde{I}_{\pm}(t+\eta)]\} d\eta. \end{aligned} \tag{4.10}$$

The first term is included in order that the form may satisfy the condition $u(0,t) = 0$; the second and third terms arise since, for an arbitrary function f,

$$(\partial/\partial t \pm \partial/\partial x)\{f(t \pm x)\} = 2f'(t \pm x), \tag{4.11}$$

$$(\partial/\partial t \pm \partial/\partial x)\{xf(t \mp x)\} = \pm f(t \mp x). \tag{4.12}$$

Substitution from (4.10) into (4.7) and identification of corresponding terms then yields the required forms for the several arbitrary functions in (4.10).

The particular integral corresponding to nonlinear magnetogasdynamic effects arises from the last terms in braces in (4.7) and is obtained directly by using the identities

$$(\partial/\partial t \pm \partial/\partial x)\{\pm xf'(t \mp x)f(t \mp x)\} = f'(t \mp x)f(t \mp x), \tag{4.13}$$

$$(\partial/\partial t \pm \partial/\partial x)\{f'(t \mp x)\hat{f}(t \pm x)\} = 2f'(t \mp x)f(t \pm x), \tag{4.14}$$

$$(\partial/\partial t \pm \partial/\partial x)\{f(t-x)f(t+x)\} = 2f'(t \pm x)f(t \mp x), \tag{4.15}$$

$$(\partial/\partial t \pm \partial/\partial x)\{f^2(t \pm x)\} = 4f'(t \pm x)f(t \pm x). \tag{4.16}$$

Without modification the form derived satisfies the boundary condition on u at $x = 0$.

Hence by adding the complementary function to these three contributions to the particular integral, the general solution to (4.7) satisfying $u = 0$ at $x = 0$ is obtained. This may be written as

$$
\begin{aligned}
u \pm \rho = {} & \pm 2f(t \mp x) + \frac{48\beta^2 D^2}{B(1+3D^2)} \left\{ \frac{T_w}{2+3^{\frac{1}{2}} - (2-3^{\frac{1}{2}}) \exp(-2\times 3^{\frac{1}{2}}Dl)} \right\} \\
& \times \{[1-\exp(-2\times 3^{\frac{1}{2}}Dl)] \sin(t\mp x) + \exp(-3^{\frac{1}{2}}Dx)(\pm 3^{\frac{1}{2}}D\cos t - \sin t) \\
& + \exp[-3^{\frac{1}{2}}D(2l-x)](\pm 3^{\frac{1}{2}}D\cos t + \sin t)\} - \frac{24\beta^2(\gamma-1)D^3}{B} \\
& \times \Bigg\{ \pm 2 \int_0^x \int^\eta \{\cosh 3^{\frac{1}{2}}D(x-\zeta)\hat{f}(t\mp\zeta) - \cosh 3^{\frac{1}{2}}D(x+\zeta-2\eta)\hat{f}(t\pm\zeta)\}\, d\zeta\, d\eta \\
& \mp \int^l [1+\exp\{-2\times 3^{\frac{1}{2}}D(l-\eta)\}] \int^\eta \Bigg[\{\cosh 3^{\frac{1}{2}}D(x-\zeta)\hat{f}(t\mp\zeta) \\
& - \cosh 3^{\frac{1}{2}}D(x+\zeta-2\eta)\hat{f}(t\pm\zeta)\} \frac{2+3^{\frac{1}{2}}-(2-3^{\frac{1}{2}})\exp(-2\times 3^{\frac{1}{2}}D\eta)}{2+3^{\frac{1}{2}}-(2-3^{\frac{1}{2}})\exp(-2\times 3^{\frac{1}{2}}Dl)} \\
& + \{\sinh 3^{\frac{1}{2}}D(x-\zeta)\hat{f}(t\mp\zeta) - \sinh 3^{\frac{1}{2}}D(x+\zeta-2\eta)\hat{f}(t\pm\zeta)\} \\
& \times \frac{2+3^{\frac{1}{2}}+(2-3^{\frac{1}{2}})\exp(-2\times 3^{\frac{1}{2}}D\eta)}{2+3^{\frac{1}{2}}-(2-3^{\frac{1}{2}})\exp(-2\times 3^{\frac{1}{2}}Dl)} \Bigg] d\zeta\, d\eta \\
& - \frac{1}{2}\int^l [1+\exp\{-2\times 3^{\frac{1}{2}}D(l-\eta)\}] \int^\eta \Bigg[\{\cosh 3^{\frac{1}{2}}D\zeta + \cosh 3^{\frac{1}{2}}D(\zeta-2\eta)\} \\
& \times \frac{2+3^{\frac{1}{2}}-(2-3^{\frac{1}{2}})\exp(-2\times 3^{\frac{1}{2}}D\eta)}{2+3^{\frac{1}{2}}-(2-3^{\frac{1}{2}})\exp(-2\times 3^{\frac{1}{2}}Dl)} - \{\sinh 3^{\frac{1}{2}}D\zeta - \sinh 3^{\frac{1}{2}}D(\zeta-2\eta)\} \\
& \times \frac{2+3^{\frac{1}{2}}+(2-3^{\frac{1}{2}})\exp(-2\times 3^{\frac{1}{2}}D\eta)}{2+3^{\frac{1}{2}}-(2-3^{\frac{1}{2}})\exp(-2\times 3^{\frac{1}{2}}Dl)} \Bigg] [\hat{f}(t\mp x+\zeta) - \hat{f}(t\mp x-\zeta)]\, d\zeta\, d\eta \\
& \pm \frac{1}{\beta^2} \int^{t\pm x} \int^\eta \cosh 3^{\frac{1}{2}}D(\eta-\zeta)\hat{f}(\zeta)\, d\zeta\, d\eta \\
& \pm \frac{1}{\beta^2} \int^{t\mp x} \int^\eta \cosh 3D(\eta-\zeta)\hat{f}(\zeta)\, d\zeta\, d\eta \mp 2x \int^x \cosh 3^{\frac{1}{2}}D(x-\eta)\hat{f}(t\mp\eta)\, d\eta \\
& + \frac{1}{3D^2}[2xf(t\mp x) \mp \hat{f}(t\pm x) \mp \hat{f}(t\mp x)] \Bigg\} + \{[3+(\gamma-2)\beta^2]xf'(t\mp x)f(t\mp x) \\
& \pm \tfrac{1}{2}[1-(\gamma-2)\beta^2][f(t-x)f(t+x) + \tfrac{1}{2}f^2(t\pm x) - f'(t\mp x)\hat{f}(t\pm x)]\}. \qquad (4.17)
\end{aligned}
$$

The arbitrary function f in this solution is determined by application of the remaining boundary condition $u(l,t) = 0$ from the set (2.28). Now we are interested in the behaviour of the solution near resonance, for which $|\Delta| \ll 1$, where $\Delta = l - N\pi$ and N is an integer. Since the driving radiation from the wall is periodic with period 2π in t, the solution is also periodic, so that

$$f(t+l-\Delta) = f(t-l+\Delta). \qquad (4.18)$$

Expanding these to first order in Δ and using the fact that, to the same order,

$$\Delta = \tan\Delta = \tan(l-N\pi) = \tan l, \qquad (4.19)$$

we may write

$$f(t+l) = f(t-l) + 2\tan l f'(t-l). \tag{4.20}$$

Similar forms may be obtained for $f'(t+l)$ and $\hat{f}(t+l)$. Thus by inserting this approximation into (4.17) after application of the boundary condition and setting $\tau = t-l$, one finds that $f(\tau)$ must satisfy the equation

$$\begin{aligned}0 = & -2\tan l f'(\tau) + \{3+(\gamma-2)\beta^2\} l f(\tau) f'(\tau) \\ & + \frac{48\beta^2 D^2\{1-\exp(-2\times 3^{\frac{1}{2}}Dl)\}T_w}{B(1+3D^2)\{2+3^{\frac{1}{2}}-(2-3^{\frac{1}{2}})\exp(-2\times 3^{\frac{1}{2}}Dl)\}}\sin\tau - \frac{12\beta^2(\gamma-1)D^3}{B} \\ & \times\left\{\int_0^{N\pi}\int^{\nu}\frac{1+\exp[-2\times 3^{\frac{1}{2}}D(N\pi-\nu)]}{2+3^{\frac{1}{2}}-(2-3^{\frac{1}{2}})\exp[-2\times 3^{\frac{1}{2}}DN\pi]}\right. \\ & \times\{[2+3^{\frac{1}{2}}-(2-3^{\frac{1}{2}})\exp(-2\times 3^{\frac{1}{2}}D\nu)][\cosh 3^{\frac{1}{2}}D(\lambda-2\nu)+\cosh 3^{\frac{1}{2}}D\lambda] \\ & +[2+3^{\frac{1}{2}}+(2-3^{\frac{1}{2}})\exp(-2\times 3^{\frac{1}{2}}D\nu)][\sinh 3^{\frac{1}{2}}D(\lambda-2\nu)-\sinh 3^{\frac{1}{2}}D\lambda]\} \\ & \times\{\hat{f}(\tau-\lambda)-\hat{f}(\tau+\lambda)\}\,d\lambda\,d\nu - 2N\pi\int^{N\pi}\cosh 3^{\frac{1}{2}}D(N\pi-\nu) \\ & \left.\times\{\hat{f}(\tau+N\pi-\nu)-\hat{f}(\tau+N\pi+\nu)\}\,d\nu + \frac{4N\pi}{3D^2}f(\tau)\right\}.\end{aligned} \tag{4.21}$$

A more convenient form of this equation is obtained if the hyperbolic functions in the double integral are written in terms of exponential functions and the entire equation integrated with respect to τ. Hence, so doing, and introducing the supplementary notation

$$\epsilon = \frac{192\beta^2 D^2 T_w}{\{3+(\gamma-2)\beta^2\} lB(1+3D^2)}\left\{\frac{1-\exp(-2\times 3^{\frac{1}{2}}Dl)}{2+3^{\frac{1}{2}}-(2-3^{\frac{1}{2}})\exp(-2\times 3^{\frac{1}{2}}Dl)}\right\}, \tag{4.22}$$

$$\delta = 12(\gamma-1)D\beta^2/\{3+(\gamma-2)\beta^2\} lB\epsilon^{\frac{1}{2}}, \tag{4.23}$$

$$r = \pi\tan l/\{3+(\gamma-2)\beta^2\} l\epsilon^{\frac{1}{2}}, \tag{4.24}$$

$$\Omega = 2\times 3^{\frac{1}{2}}/\{2+3^{\frac{1}{2}}-(2-3^{\frac{1}{2}})\exp(-2\times 3^{\frac{1}{2}}Dl)\}, \tag{4.25}$$

$$F(\tau) = f(\tau)/\epsilon^{\frac{1}{2}}, \tag{4.26}$$

it follows that $F(\tau)$ must satisfy the nonlinear integral equation

$$\begin{aligned}C+\tfrac{1}{2}\cos\tau = & \{F(\tau)-2r/\pi\}^2 \\ & -2\delta\left\{\tfrac{4}{3}N\pi\int^{\tau}F(\tau)\,d\tau - 2N\pi D^2\int^{N\pi}\int^{\tau}\cosh[3^{\frac{1}{2}}D(N\pi-\nu)]\right. \\ & \times\{\hat{F}(\tau+N\pi-\nu)-\hat{F}(\tau+N\pi+\nu)\}\,d\tau\,d\nu \\ & +\Omega D^2\int_0^{N\pi}\int^{\nu}\int^{\tau}\{\exp(-3^{\frac{1}{2}}D\lambda)+\exp[-3^{\frac{1}{2}}D(2N\pi-\lambda)] \\ & +\exp[-3^{\frac{1}{2}}D(2\nu-\lambda)]+\exp[-3^{\frac{1}{2}}D(2N\pi-2\nu+\lambda)]\} \\ & \left.\times\{\hat{F}(\tau-\lambda)-\hat{F}(\tau+\lambda)\}\,d\tau\,d\lambda\,d\nu\right\}.\end{aligned} \tag{4.27}$$

The value of constant C is determined by using the fact that since $F(\tau)$ is periodic its mean value over a period is zero.

It is worthy of note that an analysis of the same problem using the exponential approximation to the exact equation of radiative transfer leads to the same fundamental integral equation governing the flow pattern. With the exponential integral $E_2(\zeta)$ in the exact formulation replaced by $m\exp(-n\zeta)$ the various parameters in (4.27) are then defined by

$$\epsilon = \frac{32mn\beta^2D^2T_w}{\{3+(\gamma-2)\beta^2\}lB(1+n^2D^2)}\{1-\exp(-2nDl)\}, \tag{4.22a}$$

$$\delta = 4mn^2(\gamma-1)D\beta^2/\{3+(\gamma-2)\beta^2\}lB\epsilon^{\frac{1}{2}}, \tag{4.23a}$$

$$r = \pi\tan l/\{3+(\gamma-2)\beta^2\}l\epsilon^{\frac{1}{2}}, \tag{4.24a}$$

$$\Omega = 1. \tag{4.25a}$$

Clearly the parameters are little different from those in the case using the differential approximation provided that one takes $m = 1$ and $n = 3^{\frac{1}{2}}$.

Now (4.27) with $\Omega = 1$ is in all respects identical with that derived by Eninger & Vincenti, which governs the corresponding problem in the absence of electromagnetic effects. In the present work these effects are accounted for by the presence of the parameter β in the definitions (4.22a)–(4.24a) for the constants ϵ, δ and r. Thus the general behaviour of oscillations near resonance in radiative magnetogasdynamics is similar to that in radiative gasdynamics. However, since the analysis of the integral equation is presented in the paper by Eninger & Vincenti it is not repeated here and the reader is referred to their report for the details.

5. Conclusions

The significance of the function F is brought out more clearly if one considers the pressure perturbation in the gas near resonance, noting at the same time that, to leading order, from (2.13) and (2.21) the perturbations of the density and magnetic field are proportional to it. The basic expression is obtained by combining (2.21) and (2.24), taking terms of leading order and substituting for u from the solution (4.17). One finds that, after some algebra and providing D is not a small fraction, the pressure perturbation is given by

$$p \simeq \gamma\epsilon^{\frac{1}{2}}\{F(t-x)+F(t+x)\}, \tag{5.1}$$

and in particular on the reflecting wall

$$p|_{x=l} \simeq 2\gamma\epsilon^{\frac{1}{2}}F(t-l). \tag{5.2}$$

First we examine the consequences of the particular choice of radiative approximation, be it differential or exponential. The effects are apparently only in the forms of the parameters as given by (4.22)–(4.25) and (4.22a)–(4.25a). One finds, after choosing $m = 1$ and $n = 3^{\frac{1}{2}}$, that

$$\frac{\epsilon_e}{\epsilon_d} = \frac{\Omega_e}{\Omega_d} = \left\{\frac{r_d}{r_e}\right\}^2 = \left\{\frac{\delta_d}{\delta_e}\right\}^2 = \frac{(2+3^{\frac{1}{2}})-(2-3^{\frac{1}{2}})\exp(-2\times 3^{\frac{1}{2}}Dl)}{2\times 3^{\frac{1}{2}}}, \tag{5.3}$$

where the subscripts e and d stand for 'exponential' and 'differential' respectively. This ratio varies monotonically with increase of the essentially positive parameter Dl from unity to an upper bound of $(2+3^{\frac{1}{2}})/2\times 3^{\frac{1}{2}} \doteqdot 1{\cdot}08$. Thus the difference between the two approximations is quite trivial and the predicted flow patterns are virtually unaffected by the particular choice of approximation used.

The essential features of the response near resonance depend upon the values of the two parameters δ and r, which are such that

$$\frac{\delta_m}{\delta_g} = \beta\left\{\frac{\gamma+1}{3-(2-\gamma)\beta^2}\right\}^{\frac{1}{2}}, \tag{5.4}$$

$$\frac{r_m}{r_g} = \frac{1}{\beta}\left\{\frac{\gamma+1}{3-(2-\gamma)\beta^2}\right\}^{\frac{1}{2}}, \tag{5.5}$$

where the subscripts m and g stand for 'magnetogasdynamic' and 'gasdynamic' respectively. Since for real gases $1 < \gamma < 2$ and β^2 is fractional the ratio $(\gamma+1)/\{3-(2-\gamma)\beta^2\}$ is always positive and less than unity. Precisely at resonance ($r = 0$) the dominant effect of increasing δ is to diminish the shock strength and this rapidly. Now since, all other quantities being unchanged, $\delta \sim 1/B$ we note that δ increases as the general level of radiative transfer within the gas increases. Hence in the magnetogasdynamic case to attain the same value of δ as in the gasdynamic case the general level of radiation must be greater. Thus the effect of the magnetic interaction is to diminish the effects of radiative transfer.

Further, for situations near resonance Eninger & Vincenti's calculations show that, for specified D, with given r a value of δ is reached at which the discontinuity disappears, so that the shock is fully dispersed, and that as r increases the value of this critical δ decreases, but more slowly. Thus, noting from (5.4) and (5.5) that for fixed β the difference between r_g and r_m is proportionally greater than the corresponding change in δ, it follows that a greater level of radiation is required in the magnetogasdynamic case before the shock is fully dispersed.

REFERENCES

Cabannes, H. 1970 *Theoretical Magnetofluiddynamics*. Academic.

Chester, W. 1964 *J. Fluid Mech.* **18**, 44.

Eninger, J. E. & Vincenti, W. G. 1973 *J. Fluid Mech.* **60**, 161.

Helliwell, J. B. 1966 *Phys. Fluids*, **9**, 1869.

Long, H. R. & Vincenti, W. G. 1967 *Phys. Fluids*, **10**, 1365.

Vincenti, W. G. & Kruger, C. H. 1965 *Introduction to Physical Gasdynamics*. Wiley.